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THE CONDITIONS FOR EVEN FORMS TO HAVE FIXED SIGN AND FOR THE STABILITY AS A WHOLE OF NON-LINEAR HOMOGENEOUS SYSTEMS*

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Sufficient conditions are obtained for forms of arbitrary even power to have have fixed sign. A new criterion is proposed for a quadratic form to have fixed sign, which has the property of a recurrent procedure. The results obtained here and the Liapunov second method are used to obtain sufficient conditions for asymptotic stability as a whole of the solution of a set of ordinary differential equations that have their right sides in the form homogeneous polynomials.

1. Suppose that in the region $G_x \subset R^n$ a continuous real function $F(x)$ of variables $(x_1, \ldots, x_n) = \mathbf{x} \in R^n$ is specified with a range of values $H_F \subset R^1$ that vanish at the origin of coordinates

$$F(0) = 0 \tag{1.1}$$

It is required to find the conditions under which the function $F(\mathbf{x})$ is positive definite, i.e.

$$F(\mathbf{x}) > 0, \quad \forall \mathbf{x} \in G_x \setminus 0 \tag{1.2}$$

For this we introduce the function $P(y)$ of the variables $(y_1, \ldots, y_m) = \mathbf{y}$ which is continuous and real, and is defined in the region $G_y \subset R^m$ with the domain of values $H_P \subset R^1$ with $H_P \supset H_F$, and is positive definite in G_y, i.e.

$$P(\mathbf{y}) > 0, \quad \forall \mathbf{y} \in G_y \setminus 0, \ P(0) = 0 \tag{1.3}$$

The variables $y_1, \ldots, y_m$ and $x_1, \ldots, x_n$ are related by some mapping

$$y_i = f(x_1, \ldots, x_n), \ i = 1, \ldots, m \tag{1.4}$$

Let the mapping (1.4) and the functions $F(\mathbf{x}), P(\mathbf{y})$ have the following properites.

A. The region of definition of the mapping (1.4) coincides with the region G_x, while the region of values is some subset G_y^* of the set G_y $(G_y^* \subset G_y)$ with origin of coordinates $\mathbf{y} = 0 \in G_y^*$, and on the set of points G_y^* the identity

$$\forall \mathbf{y} \in G_y^*, \quad \exists x \in G_x\colon \ F(\mathbf{x}) = P(f(\mathbf{x})) \tag{1.5}$$

is satisfied, i.e. to each point $\mathbf{y} \in G_y^*$ there corresponds at least one point $\mathbf{x} \in G_x$ at which the value of the function $F(\mathbf{x})$ is the same as the value of the function $P(\mathbf{y})$ in mapping (1.4).

B. The equations $y_1 = 0, \ldots, y_m = 0$ are simultaneously satisfied when and only when all the variables $x_1, \ldots, x_n$, simultaneously vanish, i.e. when at least one coordinate $x_j \neq 0$; then necessarily we have the coordinate $y_i \neq 0$.

C. If the function $P(\mathbf{y})$ is positive definite in G_y^*, it is positive definite throughout G_y.

Theorem 1. When properties A and B are satisfied, for the function $F(\mathbf{x})$ to be positive-definite in region G_x, it is sufficient that the function $P(\mathbf{y})$ be positive definite in region G_y.

Proof. Let the function $P(\mathbf{y})$ be positive definite in G_y, i.e. conditions (1.3) are satisfied. By virtue of property A we have $G_y^* \subset G_y$ and the point $y = 0 \in G_y^*$. Hence $P(\mathbf{y})$ is positive definite in G_y^*. Moreover, by virtue of the same property A the identity (1.5) is satisfied in G_y^*. Then $F(\mathbf{x}) > 0, \ \forall \mathbf{x} \in G_x$, with the exception of the prototype of point $\mathbf{y} = \mathbf{0}$ in the region G_x, i.e. condition (1.2) is satisfied. But $P(0) = 0$ hence from property B it follows that the function $P(\mathbf{y})$ vanishes only when condition $x_j = 0, \ \forall j = 1, \ldots, n$ is satisfied. Consequently, the prototype of point $\mathbf{y} = \mathbf{0}$ in G_x is the unique point $\mathbf{x} = \mathbf{0}$. Hence condition (1.1) is satisfied for the function $F(x)$, and it is positive definite in G_x.

Theorem 2. When the properties A, B, and C are satisfied, for the function $F(x)$ to be positive-definite in region G_x it is necessary and sufficient that the function $P(\mathbf{y})$ be positive definite in region G_y.

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Proof. Necessity. Let the function $F(\mathbf{x})$ be positive definite in G_x i.e. conditions (1.1) and (1.2) are satisfied. Then by virtue of property A of mapping (1.4) to each point $\mathbf{y} \in G_y^*$ there corresponds at least one point $\mathbf{x} \in G_x$, and at these points $\mathbf{x}$ and $\mathbf{y}$ the identity (1.5) is satisfied. But at each point $\mathbf{x} \in G_x$ $(\mathbf{x} \neq 0)$ the function $F(\mathbf{x}) > 0$ and, consequently, in conformity with identity (1.5) the function $P(\mathbf{y}) > 0$, $\forall \mathbf{y} \in G_y^* \setminus 0$ also. And since property B holds, the second of conditions (1.3), is satisfied, i.e. at the point $\mathbf{y} = \mathbf{0}$ of space R^m the function $P(\mathbf{y}) = \mathbf{0}$. Thus the function $P(\mathbf{y})$ is positive definite in G_y^*. From property C it follows that the function $P(\mathbf{y})$ is positive definite throughout region G_y, i.e. conditions (1.3) are satisfied.

The proof of the sufficiency of Theorem 2 is the same as the proof of Theorem 1.

2. Let us use the results obtained above to find the sufficient conditions for a form of powere $2s$ (s is a positive integer $s > 1$) to have fixed sign, i.e. the function

$$F(\mathbf{x}) = \sum_{i_1=1}^{n} \sum_{i_2=i_1}^{n} \cdots \sum_{i_{2s}=i_{2s-1}}^{n} A_{i_1 i_2 \ldots i_{2s}} x_{i_1} x_{i_2} \ldots x_{i_{2s}}, \quad \mathbf{x} = (x_1, \ldots, x_n) \in R^n \tag{2.1}$$

where $A_{i_1 i_2 \ldots i_{2s}}$ are real numbers.

In form (2.1) similar terms are reduced and arranged in a lexicographic order.

Consider the following mapping:

$$y_1 = x_1^s, \ y_2 = x_1^{s-1} x_2, \ y_3 = x_1^{s-1} x_3, \quad \ldots, \ y_m = x_n^s \tag{2.2}$$

It is known that the elements $y_1, \ldots, y_m$ of the mapping (2.2) are linearly independent, and the vectors $y = (y_1, \ldots, y_m)$ consistute a linear space of dimension m /1/, which we denote by Y^m (Y^m is the model of space R^m).

The quadratic form in space Y^m has the form

$$P(\mathbf{y}) = \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} B_{j_1 j_2} y_{j_1} y_{j_2}, \quad B_{j_1 j_2} = B_{j_2 j_1} \tag{2.3}$$

Substituting $y_1, \ldots, y_m$ from mapping (2.2) into the quadratic form (2.3) and collecting like terms, we obtain formula (2.1) of power $2s$, i.e. we have the identity

$$\forall \mathbf{y} \in G_y^*, \ \exists \mathbf{x} \in R^n: \quad \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} B_{j_1 j_2} y_{j_1} y_{j_2} = \sum_{i_1=1}^{n} \sum_{i_2=i_1}^{n} \cdots \sum_{i_{2s}=i_{2s-1}}^{n} A_{i_1 i_2 \ldots i_{2s}} x_{i_1} x_{i_2} \ldots x_{i_{2s}} \tag{2.4}$$

where G_y^* is the region of the mapping (2.2) in the space Y^m $(G_y^* \subset Y^m)$ and the point $\mathbf{y} = \mathbf{0} \in G_y^*$.

Consequently, property A is satisfied for the mapping (2.2) for form $F(x_1, \ldots, x_n)$ (2.1), and for the quadratic form $P(y_1, \ldots, y_m)$ (2.3).

For the mapping (2.2) the property B is also satisfied. Indeed, if some coordinate $x_j \neq 0$, then in accordance with the mapping (2.2) the coordinate $y_i = x_j^s \neq 0$.

Thus, in conformity with Theorem 1, for a form of even power (2.1) to be positive definite it is sufficient that the quadratic form (2.3) be positive definite. Writing some criterion for the quadratic form (2.3) to have fixed sign and expressing in it the coefficients $B_{j_1 j_2}$ of quadratic form (2.3) in terms of the coefficients $A_{i_1 i_2 \ldots i_{2s}}$ of a form of even power (2.1) in conformity with identity (2.4), we obtain the required conditions for a form of even power (2.1) to have fixed sign.

Let us show by means of an example how the coefficients of the form of even power (2.1) are used to determine the coefficients of the quadratic form (2.3).

Example 1. Suppose we are given a form of the fourth power of two variables with constant coefficients

$$F(x_1, x_2) = A_{1111} x_1^4 + A_{1112} x_1^3 x_2 + A_{1122} x_1^2 x_2^2 + A_{1222} x_1 x_2^3 + A_{2222} x_2^4 \tag{2.5}$$

We introduce the mapping

$$y_1 = x_1^2, \quad y_2 = x_1 x_2, \quad y_3 = x_2^2 \tag{2.6}$$

The vectors $y = (y_1, y_2, y_3)$ constitute the linear space Y^3 /1/ in which the form is defined by

$$P(y_1, y_2, y_3) = B_{11} y_1^2 + 2B_{12} y_1 y_2 + 2B_{13} y_1 y_3 + B_{22} y_2^2 + 2B_{23} y_2 y_3 + B_{33} y_3^2 \tag{2.7}$$

To represent a form of the fourth power (2.5) as a quadratic form (2.7) it is necessary to add to the function (2.5) the term $\alpha_{1122} x_1^2 x_2^2$ and subtract $\alpha_{1122} (x_1 x_2)^2$. As a result, the function (2.5) takes the form

$$F(x_1, x_2) = A_{1111} (x_1^2)^2 + A_{1112} x_1^2 x_1 x_2 + \alpha_{1122} x_1^2 x_2^2 + (A_{1122} - \alpha_{1122}) (x_1 x_2)^2 + A_{1222} x_1 x_2 x_2^2 + A_{2222} (x_2^2)^2 \tag{2.8}$$

Equating functions (2.7) and (2.8), taking into account mapping (2.6), and equating the coefficients of like terms of these functions, we obtain the coefficients of the quadratic form (2.7)

$$B_{11} = A_{1111}, \quad B_{12} = {}^1/_2 A_{1112}, \quad B_{13} = {}^1/_2 \alpha_{1122} \\ B_{22} = A_{1122} - \alpha_{1122}, \quad B_{23} = {}^1/_2 A_{1222}, \quad B_{33} = A_{2222} \tag{2.9}$$

Proceeding in a similar manner with the form of even power (2.1), we obtain the coefficients $B_{j_1 j_2}$ of the quadratic form (2.3), which are thus functions of the coefficients $A_{i_1 i_2 \ldots i_{2s}}$ and of the real ancilliary numbers $\alpha_{i_1 i_2 \ldots i_{2s}}$.

3. The Sylvester criterion /1/ is generally used to check that the quadratic forms are of fixed sign. Here, using Theorem 2, we obtain a new criterion for the quadratic form to have fixed sign, which is a fairly simple recurrent calculation procedure.

Let the quadratic form be specified with constant real coefficients

$$\Phi(\mathbf{x}) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} A_{i_1 i_2} x_{i_1} x_{i_2}, \quad A_{i_1 i_2} = A_{i_2 i_1} \tag{3.1}$$

Theorem 3. For the quadratic form (3.1) to be positive definite, it is necessary and sufficient that real numbers a_{ij} are found, which are defined in terms of the coefficients $A_{i_1 i_2}$ of the quadratic form (3.1) by the recurrent formula

$$a_{ij} = \frac{1}{a_{ii}} \left(A_{ij} - \sum_{k=1}^{i-1} a_{ki} a_{kj} \right); \quad i = 1, \ldots, n \\ j = i, i+1, \ldots, n; \quad i > k \geqslant 1 \tag{3.2}$$

and satisfy the conditions

$$a_{ii} \neq 0, \ \forall i = 1, \ldots, n \tag{3.3}$$

i.e. the diagonal elements of the triangular matrix $\| a_{ij} \|_1^n$ are non-zero.

Proof. In a real space R^n, non-degenerate to a real linear transformation, any quadratic form is reducible to a normal form /1/. We shall show that then properties A, B, and C defined in Sect.1 are satisfied. We shall use Theorem 2.

We consider the quadratic form (3.1) to be the function $F(\mathbf{x})$ that appears in Theorem 2, and consider as the function $P(y)$ the normal form of that quadratic form. In this case $m = n$. The non-degenerate real linear transformation $\mathbf{y} = A\mathbf{x}$, where A is an $(n \times n)$ matrix of numbers, is a special case of mapping (1.4), when properties A, B, and C hold. Indeed, the transformation $\mathbf{y} = A\mathbf{x}$ being one-to-one is non-degenerate. Hence the region of definition and of values of this transformation coincide with R^n, and identity (1.5) is satisfied throughout the space R^n.

Thus it follows from Theorem 2 that for the quadratic form (3.1) to be positive definite it is necessary and sufficient that the normal form of that quadratic form be positive definite, i.e. it is necessary and sufficient that the following equation is satisfied:

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} A_{i_1 i_2} x_{i_1} x_{i_2} = \sum_{i=1}^{n} y_i^2 \tag{3.4}$$

where

$$y_1 = a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \\ y_2 = a_{22} x_2 + \ldots + a_{2n} x_n; \ \ldots; \ y_n = a_{nn} x_n \tag{3.5}$$

all a_{ij} are real numbers, and condition (3.3) is satisfied.

We substitute the linear functions (3.5) into the right side of Eq.(3.4) and equate the coefficients of like terms on the right and left sides of that equation. We obtain the following set of algebraic equation

$$\sum_{k=1}^{i} a_{ki} a_{kj} = A_{ij}; \quad i = 1, \ldots, \ n, \ j = i, i+1, \ldots, n$$

whose solution is constructed consecutively by the recurrent formula (3.2) beginning with the first equation, subject to condition (3.3). The theorem is proved.

The new criterion for the quadratic form to be of fixed sign obtained here does not require a calculation of determinants, as does the Sylvester criterion, and is convenient for programming and for use on a computer. Moreover, the selection of coefficients of the quadratic form (3.1), for which the latter becomes positive definite, is considerably easier. This follows from the form of the recurrent formula (3.2).

4. Let us use the criterion obtained for the quadratic form to be of fixed sign to obtain the sufficient conditions for a form of even power (2.1) to be of fixed sign. For this we first prove the corollary of Theorem 1.

Corollary 1. For a form of even power (2.1) to be positive definite it is sufficient that real numbers $\alpha_{i_1 i_2 \ldots i_{2s}}$ and a_{ij} exist which satisfy the recurrent formula

$$a_{ij} = \frac{1}{a_{ii}} \left[B_{ij}(A_{i_1 i_2 \ldots i_{2s}}, \alpha_{i_1 i_2 \ldots i_{2s}}) - \sum_{k=1}^{i-1} a_{ki} a_{kj} \right] \tag{4.1}$$
$$i = 1, \ldots, m; \quad j = i, i+1, \ldots, m$$

and the condition

$$a_{ii} \neq 0, \ \forall i = 1, \ldots, m \tag{4.2}$$

where m is the number of independent variables of the quadratic form (2.3) and $B_{ij}(A_{i_1 i_2 \ldots i_{2s}}, \alpha_{i_1 i_2 \ldots i_{2s}})$ are the coefficients of the quadratic form (2.3) that depend on the coefficients $A_{i_1 i_2 \ldots i_{2s}}$ of the form of even power (2.1), and on the numbers $\alpha_{i_1 i_2 \ldots i_{2s}}$ (2.9).

Proof. Let the conditions of Corollary 1 be satisfied. Then by the fixed-sign criterion proved in Theorem 3, the quadratic form $P(\mathbf{y})$ (2.3) will be positive definite in the space Y^m. It is connected to the form of even power (2.1) by the mapping (2.2) for which properties A and B hold. Then on the basis of Theorem 1 in which $G_x = R^n$, $G_y = Y^m$, the even-power form (2.1) is positive definite in R^n.

Example 2. The sufficient conditions for a fourth-power form (2.5) to be of fixed sign are in accordance with Corollary 1 equivalent to the existence of the following numbers a_{ij} and α_{1122}:

$$a_{11} = \pm A_{1111}^{1/2}, \quad a_{12} = \frac{1}{a_{11}} A_{1112}, \quad a_{13} = \frac{1}{a_{11}} \alpha_{1122} \tag{4.3}$$
$$a_{22} = \pm (A_{1122} - \alpha_{1122} - a_{12}^2)^{1/2}, \quad a_{23} = \frac{1}{a_{22}} (A_{1222} - a_{12} a_{13})$$
$$a_{33} = \pm (A_{2222} - a_{13}^2 - a_{23}^2)^{1/2}$$

where the number α_{1122} is selected so that the conditions

$$a_{11} \neq 0, \quad a_{22} \neq 0, \quad a_{33} \neq 0 \tag{4.4}$$

be satisfied.

Formulas (4.3) are a special case of the recurrent formula (4.1), and are obtained by applying the fixed-sign criterion (3.2) and (3.3), proved in Theorem 3, to the quadratic form (2.8) with coefficients (2.9).

We will obtain another form of sufficient conditions for a form of even power to be of fixed sign, without using the ancilliary numbers $\alpha_{i_1 i_2 \ldots i_{2s}}$. To do this we introduce an upper triangular matrix $(m \times m)$ of real numbers with non-zero diagonal elements

$$\left\| \begin{matrix} b_{\underbrace{11\ldots1}_{s}1} & b_{11\ldots12} & \cdots & b_{1n\ldots nn} \\ 0 & b_{21\ldots12} & \cdots & b_{2n\ldots nn} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{mn\ldots nn} \end{matrix} \right\| \tag{4.5}$$

$$b_{\underbrace{11\ldots11}_{s}} \neq 0, \quad b_{\underbrace{21\ldots12}_{s}} \neq 0, \ \ldots, \ b_{\underbrace{mn\ldots nn}_{s}} \neq 0 \tag{4.6}$$

We also introduce the system of algebraic equations

$$\Sigma \sum_{\beta=1}^{m} b_{\beta i_1 \ldots i_s} b_{\beta i_{s+1} \ldots i_{2s}} = A_{i_1 i_2 \ldots i_{2s}}; \quad i_1 = 1, \ldots, n \tag{4.7}$$
$$i_2 = i_1, i_1 + 1, \ldots, n; \ \ldots; \ i_{2s} = i_{2s-1}, \ldots, n$$

where Σ is the symbol of summation over permutations of those indices $i_1, i_2, \ldots, i_{2s}$, for which the following conditions are satisfied:

$$i_1 < i_2 < \cdots < i_s, \quad i_{s+1} < i_{s+2} < \cdots < i_{2s}$$

Corollary 2. The existence of a real solution $b_{\beta i_1 \ldots i_s}$ of the system of algebraic equations (4.7) with conditions (4.6), is a sufficient condition for the even-power form (2.1) to be of fixed sign.

Proof. Let a real solution of the system of algebraic equations (4.7) exist under conditions (4.6). Then for the even-power form (2.1) the following representation holds:

$$\sum_{i_1=1}^{n} \sum_{i_2=i_1}^{n} \cdots \sum_{i_{2s}=i_{2s-1}}^{n} A_{i_1 i_2 \ldots i_{2s}} x_{i_1} x_{i_2} \ldots x_{i_{2s}} = \sum_{i=1}^{m} \varphi_i^2 \tag{4.8}$$

$$\varphi_1 = b_{11\ldots11}x_1^s + b_{11\ldots12}x_1^{s-1}x_2 + \ldots + b_{1n\ldots nn}x_n^s$$
$$\varphi_2 = b_{21\ldots12}x_1^{s-1}x_2 + \ldots + b_{2n\ldots nn}x_n^s, \ldots, \varphi_m = b_{mn\ldots nn}x_n^s \qquad (4.9)$$

Indeed, by subsituting forms of power s (4.9) into the right side of Eq.(4.8) and equating the coefficients of like terms on the right and left sides of that equation, we obtain system (4.7).

Taking identity (2.4) into account and the coordinates y_k of the mapping (2.2), as well as changing the designation of coefficients in forms (4.9) from $b_{\beta i_1\ldots i_s}$ to a_{ij}, we can write Eq.(4.8) and condition (4.6) in the form

$$\sum_{j_1=1}^{m}\sum_{j_2=1}^{m} B_{j_1j_2}y_{j_1}y_{j_2} = \sum_{i=1}^{m}\varphi_i^2 \qquad (4.10)$$

$$a_{ii} \neq 0, \ \forall i = 1, \ldots, n \qquad (4.11)$$

In accordance with Theorem 3 the right side of Eq.(4.10) is a positive definite quadratic form in the space Y^m, since by assumption all numbers a_{ij} exist under condition (4.11) and, respectively, the numbers $b_{\beta i_1\ldots i_s}$ exist under condition (4.6). Then in accordance with Eq.(4.8) from Theorem 1 it follows that the even-power form (2.1) is positive definite. The corollary is proved.

Example 3. Let us write the special form of Eq.(4.8) for the fourth-power form (2.5), taking into account the mapping (2.6) and condition (4.7)

$$A_{1111}x_1^4 + A_{1112}x_1^3x_2 + A_{1122}x_1^2x_2^2 + A_{1222}x_1x_2^3 + A_{2222}x_2^4 = (b_{111}x_1^2 + b_{112}x_1x_2 + b_{122}x_2^2)^2 + (b_{212}x_1x_2 + b_{222}x_2^2)^2 + (b_{322}x_2^2)^2 \qquad (4.12)$$

$$b_{111} \neq 0, \quad b_{212} \neq 0, \quad b_{322} \neq 0 \qquad (4.13)$$

All remaining coefficients $b_{111}, b_{112}, \ldots, b_{322}$ are arbitrary real numbers.

Equating the coefficients of like terms on the left and right sides of Eq.(4.12), we obtain a system of algebraic equations which is a special form of algebraic equations (4.7), in which the number of unknowns b_{ijk} exceeds the number of equations.

We will solve system (4.7) for this example successively

$$b_{111} = \pm A_{1111}^{1/2}, \quad b_{112} = \frac{1}{2b_{111}}A_{1112}, \quad b_{122} = \frac{1}{2b_{111}}(A_{1122} - b_{112}^2 - b_{212}^2)$$
$$b_{222} = \frac{1}{2b_{212}}(A_{1222} - 2b_{112}b_{122}), \quad b_{322} = \pm(A_{2222} - b_{122}^2 - b_{222}^2)^{1/2} \qquad (4.14)$$

where the number b_{212} is specified so as not to violate condition (4.13).

In accordance with Corollary 2 the existence of the real solution (4.14) of system (4.7) with condition (4.13) is sufficient for form (2.5) to be positive definite.

If we use other criteria for the quadratic form to be of fixed sign, we obtain the sufficient conditions for the fourth-power form (2.1) to be of fixed sign that differ from the conditions obtained herein Corollaries 1 and 2. Thus by applying the Sylvester criterion /1/ to the quadratic form (2.3), we obtain the sufficient conditions for the even-power form (2.1) in the shape of inequalities for the principal minors of the matrix of quadratic form (2.3).

The conditions for even forms to be of fixed sign are convenient for applications, since they are directly expressed in terms of coefficients of that form. For a wide a class of forms, for which representation (4.8) holds, these conditions are necessary and sufficient. The necessity follows from Theorem 3.

When the form cannot be represented by (4.8) or the proposed conditions do not result in a solution of the problem, the criterion for higher forms to be of fixed sign may be used. This criterion was obtained in /2/, where it was proposed to verify the positiveness of specific forms at points of possible extremem of the form, which lie on a hypersphere with centre at the origin of coordinates. The problem then reduces to solving a system of non-linear algebraic equations, whose number is equal to the number of unknown variables, and the order of each equation exceeds by unity the power of the specified form.

5. Let us use the results obtained here to derive the sufficient conditions of asymptotic stability as a whole of the zero solution of a system of ordinary differential equations whose right side is in the form of polynomials of odd power with constant coefficients

$$\frac{dx_\alpha}{dt} = \sum_{i_1=1}^{n}\sum_{i_2=i_1}^{n}\cdots\sum_{i_{2s-1}=i_{2s-2}}^{n} a_{\alpha i_1 i_2\ldots i_{2s-1}}x_{i_1}x_{i_2}\ldots x_{i_{2s-1}}, \quad \alpha = 1, \ldots, n \qquad (5.1)$$

In the homogeneous polynomials on the right side of the equation similar terms are given and arranged in lexicographic order.

We will use the Barbashin-Krasovskii theorem on the asymptotic stability as a whole /3/. The Liapunov function is sought in the set of all negative-definite forms with constant real coefficients

$$v(\mathbf{x}) = -\frac{1}{2}\sum_{k_1=1}^{n}\Big(\sum_{k_2=k_1}^{n} p_{k_1k_2}x_{k_2}\Big)^2 \tag{5.2}$$

$$p_{k_1k_1} \neq 0, \quad \forall k_1 = 1, \ldots, n \tag{5.3}$$

By virtue of system (5.1) the derivative with respect to time t of the quadratic form is written as follows:

$$\frac{dv}{dt} = \sum_{\alpha=1}^{n}\frac{\partial v}{\partial x_\alpha}\frac{dx_\alpha}{dt} = -\sum_{\alpha=1}^{n}\sum_{k_1=1}^{n}\sum_{k_2=k_1}^{n}\sum_{i_1=1}^{n}\cdots\sum_{i_{2s-1}=i_{2s-2}}^{n} p_{k_1\alpha}p_{k_1k_2}a_{\alpha i_1\ldots i_{2s-1}}x_{i_1}\ldots x_{i_{2s-1}}x_{k_2} \tag{5.4}$$

Thus the derivative dv/dt (5.4) is a form of power $2s$ (2.1), where the coefficients $A_{i_1i_2\ldots i_{2s}}$ are given by the formulas

$$A_{i_1i_2\ldots i_{2s}} = -\sum{}'\sum_{\alpha=1}^{n}\sum_{k_1=1}^{\alpha} p_{k_1i_{2s}}p_{k_1\alpha}a_{\alpha i_1\ldots i_{2s-1}};$$
$$i_1 = 1, \ldots, n,\ i_2 = i_1, i_1+1, \ldots, n, \ldots, i_{2s} = i_{2s-1}, \ldots, n \tag{5.5}$$

where the symbol Σ' denotes summation over those permutations of indices $i_1, \ldots, i_{2s}$, that satisfy the conditions $i_1 < i_2 < \ldots < i_{2s-1},\ k_1 < i_{2s}$.

Formula (5.5) was obtained by equating the coefficients of like terms of the derivative dv/dt (5.5) and of form of power $2s$ (2.1). It is usually convenient to write in formulas (5.5), first, the coefficients $A_{i_1i_2\ldots i_{2s}}$ and then the terms $p_{k_1i_{2s}}$, $a_{\alpha i_1\ldots i_{2s-1}}$, that correspond to selected indices $i_1, i_2, \ldots, i_{2s}$.

For forms of even power the sufficient conditions for positive definitness are contained in Corollaries 1 and 2. By applying them to the derivative dv/dt (5.4), we obtain the required conditions of asymptotic stability as a whole of the zeroth solution of system (5.1).

As a preliminary, we will convert the derivative dv/dt (5.4) that is equal to the form of power $2s$ (2.1) into the quadratic form (2.3), using the real numbers $\alpha_{i_1i_2\ldots i_{2s}}$, as was done in Example 1 with the form (2.5). In this case the coefficients $B_{j_1j_2}$ of the quadratic form (2.3) depend on the variables $p_{k_1k_2}$, on the coefficients $a_{\alpha i_1\ldots i_{2s-1}}$ of system (5.1), and on the ancilliary numbers $\alpha_{i_1i_2\ldots i_{2s}}$.

$$B_{j_1j_2} = B_{j_1j_2}(p_{k_1k_2}, a_{\alpha i_1\ldots i_{2s-1}}, \alpha_{i_1i_2\ldots i_{2s}}),\quad j_1, j_2 = 1, \ldots, m \tag{5.6}$$

Theorem 4. **For the asymptotic stability as a whole of the zero solution of system (5.1) it is sufficient that real numbers a_{ij}, $p_{k_1k_2}$ and $\alpha_{i_1i_2\ldots i_{2s}}$, exist which satisfy the recurrent formula**

$$a_{ij} = \frac{1}{a_{ii}}\Big[B_{ij}(p_{k_1k_2}, a_{\alpha i_1\ldots i_{2s-1}}, \alpha_{i_1i_2\ldots i_{2s}}) - \sum_{k=1}^{i-1}a_{ki}a_{kj}\Big] \tag{5.7}$$
$$i = 1, \ldots, m,\ j = i, i+1, \ldots, m$$

with condition (4.2). Here $B_{ij}(p_{k_1k_2}, a_{\alpha i_1\ldots i_{2s-1}}, \alpha_{i_1i_2\ldots i_{2s}})$ are the coefficients (5.6) of quadratic form (2.3), which is obtained from the derivative dv/dt (5.4) by using the transformation (2.2).

Proof. Suppose numbers $p_{k_1k_2}$, $\alpha_{i_1i_2\ldots i_{2s}}$ exist that satisfy formula (5.7) and condition (4.2). Then in accordance with the Corollary 1 the derivative dv/dt (5.4) that represents a form of power $2s$ (2.1) is positive definite. According to the Barbashin-Krasovskii theorem /3/ the positive definiteness of the derivative dv/dt is sufficient for the asymptotic stability as a whole of the zeroth solution of system (5.1). The theorem is proved.

We introduce the system of algebraic equations

$$\sum\sum_{\beta=1}^{m} b_{\beta i_1\ldots i_s}b_{\beta i_{s+1}\ldots i_{2s}} + \sum{}'\sum_{\alpha=1}^{n}\sum_{k_1=1}^{\alpha} a_{\alpha i_1\ldots i_{2s-1}}p_{k_1i_{2s}}p_{k_1\alpha} = 0$$
$$i_1 = 1, \ldots, n,\ i_2 = i_1, i_1+1, \ldots, n, \ldots, i_{2s} = i_{2s-1}, \ldots, n \tag{5.8}$$

where Σ, Σ' are symbols defined in (4.7) and (5.5). Equations (5.8) are obtained from Eqs. (4.7) by replacing the coefficients $A_{i_1i_2\ldots i_{2s}}$ by their values in (5.5).

Theorem 5. For the asymptotic stability as a whole of the zeroth solution of system (5.1) it is sufficient that the system of algebraic equations (5.8) has a real solution $p_{k_1k_2}$ and $b_{\beta i_1\ldots i_s}$ that satisfy conditions (4.6) and (5.3).

Proof. Let a real solution exist of the **system** of algebraic equations (5.8) in the unknowns $p_{k_1k_2}$ and $b_{\beta i_1\ldots i_s}$ under conditions (4.6) and (5.3). Taking (5.5) into account, this is equivalent to the existence of a real solution of Eqs.(4.7). Then from Crollary 2 it follows that the derivative dv/dt (5.4), equal to a form of power $2s$ (2.1), is positive definite. The positive definiteness of the derivative dv/dt (5.4), according to the Barbashin-Krasovksii theorem mentioned above, is sufficient for the asymptotic stability as a whole of the zeroth solution of system (5.1). The theorem is proved.

Example 4. Let the determine the conditions of asymptotic stability as a whole of the zeroth solution of system

$$\frac{dx_1}{dt} = a_{1111}x_1^3 + a_{1112}x_1^2x_2 + a_{1122}x_1x_2^2 + a_{1222}x_2^3 \tag{5.9}$$
$$\frac{dx_2}{dt} = a_{2111}x_1^3 + a_{2112}x_1^2x_2 + a_{2122}x_1x_2^2 + a_{2222}x_2^3$$

The derivative with respect to time t of the quadratic form with constant real coefficients

$$v(x_1, x_2) = -{}^1/_2\,[(p_{11}x_1 + p_{12}x_2)^2 + (p_{22}x_2)^2],\quad p_{11} \neq 0,\quad p_{22} \neq 0$$

is, by virtue of system (5.9), a fourth-power form of the variables x_1, x_2, i.e.

$$\frac{dv}{dt} = A_{1111}x_1^4 + A_{1112}x_1^3x_2 + A_{1122}x_1^2x_2^2 + A_{1222}x_1x_2^3 + A_{2222}x_2^4 \tag{5.10}$$

$$A_{1111} = -p_{11}^2a_{1111} - p_{11}p_{12}a_{2111} \tag{5.11}$$
$$A_{1112} = -p_{11}^2a_{1112} - p_{11}p_{12}a_{1111} - p_{11}p_{12}a_{2112} - (p_{12}^2 + p_{22}^2)\,a_{2111}$$
$$A_{1122} = -p_{11}^2a_{1122} - p_{11}p_{12}a_{1112} - p_{11}p_{12}a_{2122} - (p_{12}^2 + p_{22}^2)\,a_{2112}$$
$$A_{1222} = -p_{11}^2a_{1222} - p_{11}p_{12}a_{1122} - p_{11}p_{12}a_{2222} - (p_{12}^2 + p_{22}^2)\,a_{2122}$$
$$A_{2222} = -p_{11}p_{12}a_{1222} - (p_{12}^2 + p_{22}^2)\,a_{2222}$$

In accordance with Theorem 4 the sufficient conditions of asymptotic stability as a whole of the zeroth solution of system (5.1) are the existence of real numbers $p_{11} \neq 0,\ p_{12},\ p_{22} \neq 0$ and numbers $\alpha_{1122},\ a_{11} \neq 0,\ a_{12},\ a_{13},\ a_{22} \neq 0,\ a_{23},\ a_{33} \neq 0$, which satisfy (4.3) and (5.11).

By Theorem 5, the sufficient conditions for asymptotic stability as a whole of the zeroth solution of (5.1) is the existence of a real solution $p_{11} \neq 0,\ p_{12},\ p_{22} \neq 0,\ b_{111} \neq 0,\ b_{112},\ b_{122},\ b_{212} \neq 0,\ b_{222},$ $b_{322} \neq 0$ of the system of algebraic equations (4.7) and (5.11).

Note that the application of the sufficient conditions for a form of even power to be of fixed sign, based on the Sylvester criterion for quadratic forms to be of fixed sign /1/, leads to the derivation of the sufficient conditions for asymptotic stability as a whole of the zeroth solution of system (5.1) in the form of inequalities for the Sylvester determinants /1/.

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ON THE STABILITY OF INVARIANT MANIFOLDS OF MECHANICAL SYSTEMS*

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The stability of degenerate invariant manifolds of steady motions of mechanical systems imbedded in one another /1/ is investigated using Liapunov's second method.

1. Statement of the problem. Problems of the separation and qualitative investigation of invariant manifolds of the steady motions of autonomous differential equations of mechanical systems

$$x_i^{\cdot} = X_i(x_1, x_2, \ldots, x_n),\ i = 1, 2, \ldots, n \tag{1.1}$$

with smooth right sides in $U \subset R^n$, generated by their first inegrals

$$V_0(x_1, \ldots, x_n) = c_0,\ V_1(x_1, \ldots, x_n) = c_1,\ \ldots,\ V_m(x_1, \ldots, x_n) = c_m \tag{1.2}$$

which are also assumed to be autonomous and smooth (even analytic) in the respective region $V \subset U \subset R^n$ are considered.

Let us set up the "complete" integral of system (1.1)

$$K = \lambda_0 V_0(x) + \lambda_1 V_1(x) + \ldots + \lambda_m V_m(x)$$

It is always possible to assume one of the quantities $\lambda_j = \text{const}$ in K to be unity. Henceforth, we will assume $\lambda_0 = 1$, since in a general consideration it is not necessary to analyse

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